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The region of analytic functions for the convergence of trigonometric interpolation[☆]

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Abstract

For $r > 0$ let $AP(D_r)$ denote the set of 2π-periodic functions which are analytic on the closed rectangle $D_r = \{z \in \mathbb{C} : 0 \le \operatorname{Re} z \le 2\pi, |\operatorname{Im} z| \le r\}$, and let $AP[0, 2\pi] = AP(D_0)$. For a positive integer n let $Z_n = \{t_{n,1}, t_{n,2}, \dots, t_{n,n}\}$ be a set of nodes in the interval $[0, 2\pi)$ such that $t_{n,1} < t_{n,2} < \cdots < t_{n,n}$, and let $T_n^Z f$ denote the trigonometric interpolation operator which interpolates $f \in AP[0, 2\pi]$ on the set Z_n. Finally, set

$$h = \limsup_{n\to\infty} \sqrt[n]{\left\| \prod_{j=1}^{n} \sin \frac{t - t_{n,j}}{2} \right\|}, \tag{$*$}$$

where $\|f\|$ denotes the maximum norm of continuous 2π-periodic function f on the interval $[0, 2\pi]$. In this paper, we define a function $r : [\frac{1}{2}, 1] \to [0, \infty)$ by $r(h)$ to be the infimum of all numbers $r > 0$ such that the sequence $(T_n^Z f)$ converges to f uniformly on $[0, 2\pi]$ for every $f \in AP(D_r)$ and every sequence of nodal sets Z_n which satisfy equation $(*)$. The main results are summarized as $\ln(4h - 1) \le r(h) \le \min\{2\ln(h + \sqrt{1 + h^2}), \ln(4h^2 + \sqrt{16h^4 - 1})\}$.

Keywords: Trigonometric interpolation; Perfect convergence; Convergence with preassigned nodal sets

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1. Introduction

For a positive integer n let $Z_n = \{ t_{n,1}, t_{n,2}, \ldots, t_{n,n}\}$ be a set of nodes in the interval $[0, 2\pi)$ such $t_{n,1} < t_{n,2} < \cdots < t_{n,n}$ and set

$$\Delta_n(t) = \prod_{j=1}^{n} \sin \frac{t - t_{n,j}}{2}. \tag{1.1}$$

For a given function $f(t)$ defined on $[0, 2\pi)$ we define the trigonometric interpolation operator T_n^Z(TIO) by

$$(T_n^Z f)(t) = \sum_{j=1}^{n} T_{n,j}^Z(t) f(t_{n,j}), \tag{1.2}$$

where

$$T_{n,j}^Z(t) = \begin{cases} \dfrac{\Delta_n(t)}{2\Delta_n'(t_{n,j})} \csc \dfrac{t - t_{n,j}}{2}, & \text{if } n \text{ is odd,} \\ \dfrac{\Delta_n(t)}{2\Delta_n'(t_{n,j})} \cot \dfrac{t - t_{n,j}}{2}, & \text{if } n \text{ is even.} \end{cases} \tag{1.3}$$

It is obvious that T_{2m-1}^Z is just the well known classical trigonometric interpolation operator in the case of odd number of nodes, and T_{2m}^Z is just the proximal interpolation operator in the case of even number of nodes given in [6] by Schönberg.

Let

$$(\delta_n^Z f)(t) = f(t) - (T_n^Z f)(t) \tag{1.4}$$

be the remainder of TIO [3].

Let $\|\cdot\|_r$ denote the Chebyshev norm (maximum norm) of 2π-periodic continuous functions on the line segment $z = x + \mathrm{i}r$ ($0 \le x \le 2\pi$, r is a fixed real number), and let $\|\cdot\| = \|\cdot\|_0$.

The trigonometric interpolation of f is said to be convergent for the sequence of the nodal sets Z_n if $\lim_{n\to\infty} \|\delta_n^Z f\| = 0$.

2. The distribution of nodal sets and the analytic region of functions

A set D in $\mathbb{C}$ is called an analytic region of a function f if f is analytic on D. It is also called the region of f simply as we always consider analytic functions.

In this paper we try to find some relations between the region of analytic functions f interpolated and the distribution of nodal sets in order to have that the trigonometric interpolation converges.

Let $AP(D_r)$ denote the set of 2π-periodic functions which are analytic on the closed rectangular domain

$$D_r = \{z : 0 \le \operatorname{Re} z \le 2\pi, |\operatorname{Im} z| \le r\}, \quad (r \ge 0). \tag{2.1}$$

Let $AP[0, 2\pi] = AP(D_0)$, that is, the set of 2π-periodic functions analytic on the real axis. And denote the upper and lower boundary of D_r by ∂D_r^+ and ∂D_r^- respectively.

Definition 2.1. The trigonometric interpolation is said to be perfectly convergent for $AP(D_r)$ ($r \geq 0$) if $\lim_{n\to\infty} \|\delta_n^Z f\| = 0$ for every sequence of nodal sets Z_n and each $f \in AP(D_r)$.

In [3,4], we obtained:

Theorem 2.1. *The trigonometric interpolation is perfectly convergent for $AP(D_r)$ if and only if $r \geq 2\ln(1+\sqrt{2})$.*

Definition 2.2. Let Z_n be a given sequence of nodal sets. The trigonometric interpolation is said to be fully convergent for Z_n if $\lim_{n\to\infty} \|\delta_n^Z f\| = 0$ for each $f \in AP[0, 2\pi]$.

In [5], we also obtained the following.

Theorem 2.2. *The trigonometric interpolation corresponding to the given sequence of nodal sets Z_n is fully convergent if and only if the sequence of the nodal sets Z_n is uniformly distributed, i.e.,*

$$\limsup_{n\to\infty} \sqrt[n]{\|\Delta_n\|} = \frac{1}{2}. \tag{2.2}$$

There are many cases of nodal sets Z_n uniformly distributed. The sequence of Chebyshev nodal sets is one such.

Example 2.1. In (1.1), we take the sequence of nodal sets Z_n: $t_{n,j} = \frac{2(j-1)\pi}{n}$ ($j = 1, 2, \ldots, n$); then $\limsup_{n\to\infty} \sqrt[n]{\|\Delta_n\|} = \frac{1}{2}$.

Theorems 2.1 and 2.2 show that there are many cases of non-uniformly distributed sequences of nodal sets Z_n. We give a concrete example [5] in the following.

Example 2.2 (*[5]*). Let $z_{n,j} = e^{it_{n,j}}$ ($j = 1, 2, \ldots, n$) where $t_{n,j}$ is given by Example 2.1. Using the Möbius transformation

$$T_\alpha(z) = \frac{z-\alpha}{1-\alpha z} \quad \text{with } 0 < \alpha < 1, \tag{2.3}$$

we get $z_{n,j}^\alpha = T_\alpha(z_{n,j})$. Take a sequence of nodal sets Z_n^α: $t_{n,j}^\alpha = \arg(z_{n,j}^\alpha)$ and define the polynomial

$$\omega_n^\alpha(z) = \prod_{k=1}^{n} (z - z_{k,n}^\alpha) \tag{2.4}$$

which has a representation as follows:

$$\omega_n^\alpha(z) = \frac{(z+\alpha)^n - (1+\alpha z)^n}{1-\alpha^n}. \tag{2.5}$$

In fact, the right hand side of the above equation is a polynomial of degree n, the coefficient of z^n is 1, and it vanishes at the points $z = z_{k,n}^\alpha$ because $(\frac{z_{k,n}^\alpha}{1+\alpha z_{k,n}^\alpha})^n = 1$.

From the representation (2.5) it is easy to see that for $|z| > 1$

$$\lim_{n\to\infty} \frac{\sqrt[n]{\omega_n^\alpha(z)}}{z} = 1 + \frac{\alpha}{z}. \tag{2.6}$$

which gives

$$\limsup_{n\to\infty,|z|=1} |\sqrt[n]{\omega_n^\alpha(z)}| = 1+\alpha. \tag{2.7}$$

By (2.7) and $(2e^{iz})^n \Delta_z^\alpha = \omega_n^\alpha(e^{iz})$ we have

$$\limsup_{n\to\infty} \sqrt[n]{\|\Delta_n^\alpha\|} = \frac{1+\alpha}{2} \tag{2.8}$$

which implies that $\{Z_n^\alpha\}$ is non-uniformly distributed.

For any sequence of nodal sets Z_n, we proved in [4]

$$\frac{1}{2} < \sqrt[n]{\|\Delta_n\|} \le 1. \tag{2.9}$$

Remark 1. From Examples 2.1 and 2.2 if we chose α close to 0 or 1, then we see that the estimate (2.9) is best possible.

We need the following result concerning Lagrange interpolation on the unit circle which was proved by Brück in [1] (Theorem 1 and its second remark).

Lemma 2.1. *Assume that f is analytic on $B_R = \{z : |z| < R\}$ $(R > 1)$; then*

$$\lim_{n\to\infty} (L_n^\alpha f)(z) = f(z), \quad (z \in G(R,\alpha)) \tag{2.10}$$

where $(L_n^\alpha f)(z)$ is the Lagrange interpolation of f by polynomials with the nodal sets of $z_{n,j}^\alpha$, and

$$G(R,\alpha) = \{z : |z+\alpha| < R-\alpha\} \cap \{z : |1+\alpha z| < R-\alpha\}. \tag{2.11}$$

On the other hand, f must be analytic on B_R when $\lim_{n\to\infty}(L_n^\alpha f)(z) = f(z)$, $(z \in G(R,\alpha))$.

The trigonometric interpolation of 2π-periodic functions corresponds to the Lagrange interpolation on the unit circle. Let $A(B_{e^r} \cap (\mathbb{C} \setminus B_{e^{-r}}))$ denote the set of functions which are analytic on $B_{e^r} \cap (\mathbb{C} \setminus B_{e^{-r}})$. If $f \in AP(D_r)$, then $f(\ln(z))$ belongs to $A(B_{e^r} \cap (\mathbb{C} \setminus B_{e^{-r}}))$. And if f belongs to $A(B_{e^r} \cap (\mathbb{C} \setminus B_{e^{-r}}))$, then $f(e^z) \in AP(D_r)$. We define a mapping $\rho : AP(D_r) \to A(B_{e^r} \cap (\mathbb{C} \setminus B_{e^{-r}}))$ such that

$$\rho(f)(z) = f(-i\ln(z)) \quad f \in AP(D_r).$$

It is easy to check that

$$(T_n^{Z^\alpha} f)(z) = (L_n^\alpha(\rho(f)))(T_\alpha(z)).$$

3. Test functions

It is difficult to directly compute the remainder of the interpolation and find the correlation between the region of functions interpolated and the distribution of nodal sets in order to have that the trigonometric interpolation converges. In this section some test functions are constructed to deal with them.

Assume that Z_n is a sequence of nodal sets. The real variable h is defined as follows:

$$h := \limsup_{n\to\infty} \sqrt[n]{\|\Delta_n\|}. \tag{3.1}$$

It is obvious that $h \in [\frac{1}{2}, 1]$ by (2.9).

Theorem 3.1. *Let h be given by (3.1) and $r \geq 2\operatorname{arsinh} h = 2\ln(h+\sqrt{1+h^2})$; then $\limsup_{n\to\infty}\|\delta_n^Z f\| = 0$ for each $f \in AP(D_r)$.*

Proof. Since D_r is a compact set, there exists R such that $R > r$ and $f \in AP(D_R)$.

The reminder of TIO can be written as follows:

$$(\delta_n^Z f)(t) = \begin{cases} \dfrac{\Delta_n(t)}{4\pi\mathrm{i}} \displaystyle\int_{\partial D_R} \dfrac{f(z)}{\Delta_n(z)} \csc\dfrac{z-t}{2}\mathrm{d}z, & \text{if } n = 2m-1, \\ \dfrac{\Delta_n(t)}{4\pi\mathrm{i}} \displaystyle\int_{\partial D_R} \dfrac{f(z)}{\Delta_n(z)} \cot\dfrac{z-t}{2}\mathrm{d}z, & \text{if } n = 2m. \end{cases} \tag{3.2}$$

Notice that for a 2π-periodic function $F(z)$ we have

$$\int_{\mathrm{i}R}^{-\mathrm{i}R} F(z)\mathrm{d}z + \int_{2\pi-\mathrm{i}R}^{2\pi+\mathrm{i}R} F(z)\mathrm{d}z = 0.$$

So

$$|(\delta_n^Z f)(t)| \leq M|\Delta_n(t)| \int_{\partial D_R^+} \frac{1}{|\Delta(z)|}|\mathrm{d}z| + M|\Delta_n(t)| \int_{\partial D_R^-} \frac{1}{|\Delta(z)|}|\mathrm{d}z|.$$

Since for $z \in \partial D_R^\pm$ we have $|\sin\frac{z-t_j}{2}| > \sinh\frac{R}{2}$ and $\frac{h}{\sinh\frac{R}{2}} < 1$, it is easy to check that

$$\|\delta_n^Z f\| < O(1)\left[\frac{h}{\sinh(\frac{R}{2})}\right]^n \to 0 \quad \text{as } n \to \infty.$$

The proof is completed. □

Remark 2. The condition $r \geq 2\operatorname{arsinh} h$ is only a sufficient condition for the convergence of trigonometric interpolation since by Theorem 2.2 the trigonometric interpolation converges for $r \geq 0$ when $\limsup_{n\to\infty}\sqrt[n]{\|\Delta_n\|} = \frac{1}{2}$. We will give more sufficient conditions in the following theorems.

Theorem 3.2. *Let h be given by (3.1) and $r \geq \ln(4h^2+\sqrt{16h^4-1})$; then $\limsup_{n\to\infty}\|\delta_n^Z f\| = 0$ for each $f \in AP(D_r)$.*

For the proof of Theorem 3.2 we present two lemmas.

Lemma 3.1. *Let h be given by (3.1) and $r > \ln(4h^2+\sqrt{16h^4-1})$; then*

$$0 < \frac{4h^2\mathrm{e}^r - 1}{\mathrm{e}^r - 4h^2} < \mathrm{e}^r. \tag{3.3}$$

Proof. The left inequality of (3.3) is obvious. Now we notice that $\lambda_0 = 4h^2 + \sqrt{16h^4-1}$ is the largest zero of the **test function**

$$\omega(\lambda) = \lambda^2 - 8h^2\lambda + 1. \tag{3.4}$$

So, $e^r > \lambda_0$ yields $\omega(e^r) > 0$ which is just the right inequality of (3.3). □

Lemma 3.2. *Assume that σ is an analytic function in the disk $\mathbb{D} = \{w \in \mathbb{C} : |w| < 1\}$ and $|\sigma(w)| \le 1$ for $w \in \mathbb{D}$. Then*

$$|\sigma(w)| \ge \frac{|\sigma(0)| - |w|}{1 - |\sigma(0)||w|}. \tag{3.5}$$

By the Schwarz lemma, its proof is easy [2].

The proof of Theorem 3.2. Let

$$h_n(z) = (-4e^{iz})^n \Delta_n^2(z), \tag{3.6}$$

which is a 2π-periodic entire function and

$$h(+i\infty) := \lim_{y\to+\infty, 0\le x\le 2\pi} h_n(z) = \exp\left(i\sum_{j=1}^{n} t_{n,j}\right), \tag{3.7}$$

where $x = \operatorname{Re} z$, $y = \operatorname{Im} z$. Let $h_n^*(z) = \rho(h_n)(z)$; then

$$h_n^*(0) = h_n(+i\infty) = \exp\left(i\sum_{j=1}^{n} t_{n,j}\right). \tag{3.8}$$

Write $M(h_n^*, s) = \max(|h_n^*(w)|, |w| = e^{-s})$; then, by the maximum modulus principle we get

$$M(h_n^*, s) \ge M(h_n^*, 0) = \|h_n\| = 4^n\|\Delta_n\|^2 \quad \text{for } s > 0. \tag{3.9}$$

We still assume $r > \ln(4h^2 + \sqrt{16h^4 - 1})$. By Lemma 3.1 we take μ such that $h < \mu$ and

$$0 < \frac{4h^2e^r - 1}{e^r - 4h^2} < \frac{4\mu^2e^r - 1}{e^r - 4\mu^2} < e^r. \tag{3.10}$$

Let

$$\sigma_n(w) = (2\mu)^{-2}\sqrt[n]{h_n^*(w)} \quad \text{for } |w| < 1. \tag{3.11}$$

Then, for n sufficiently large we may quote Lemma 3.2 and get

$$|\sigma_n(w)| \ge \frac{e^r - 4\mu^2}{4\mu^2e^r - 1} \quad \text{for } |w| = e^{-r}. \tag{3.12}$$

That is

$$\sqrt[n]{\|\Delta_n^{-1}\|_r} \le \mu^{-1}e^{-r/2}\sqrt{\frac{4\mu^2e^r - 1}{e^r - 4\mu^2}}. \tag{3.13}$$

By (3.2), (3.11) and (3.13),

$$\|\delta_n^Z f\| \le O(1)\left[\frac{4\mu^2e^r - 1}{e^r(e^r - 4\mu^2)}\right]^{\frac{n}{2}} \to 0 \quad \text{as } n \to \infty. \quad \square$$

Remark 3. By (3.13), $\limsup_{n\to\infty}\sqrt[n]{\|\Delta_n^{-1}\|_r} \le \mu^{-1}e^{-r/2}\sqrt{\frac{4\mu^2 e^r-1}{e^r-4\mu^2}}$. Let μ tend to 1/2; we have $\limsup_{n\to\infty}\sqrt[n]{\|\Delta_n^{-1}\|_r} = 2e^{-\frac{r}{2}}$ and $\limsup_{n\to\infty}\sqrt[n]{\|\Delta_n\|_r} = \frac{1}{2}e^{\frac{r}{2}}$, which are just some results in [5].

Now we introduce a new **test function**

$$\Omega(\lambda) = \lambda^3 - 2h\lambda^2 - 2h\lambda + 1. \tag{3.14}$$

In the same way as above we have the following result.

Lemma 3.3. *Assume that Ω is given by (3.14). Then there exists a largest real zero λ_0 of Ω and $\lambda_0 \ge \sqrt{2h}$. If $r > \ln\lambda_0$, then*

$$0 < \frac{2he^r - 1}{e^r - 2h} < e^{\frac{r}{2}}. \tag{3.15}$$

Proof. It may be directly verified that $\Omega(\sqrt{2h}) = 1 - 4h^2 \le 0$ since $h \ge \frac{1}{2}$, and noticing that $\Omega(+\infty) = +\infty$. Thus there exists λ_0 and $\lambda_0 \ge \sqrt{2h}$. By $e^r > \lambda_0^2 \ge 2h$ we get the left inequality of (3.15). Again from $e^{\frac{r}{2}} > \lambda_0$ we know that $\Omega(e^{\frac{r}{2}}) > 0$, which is just the inequality of (3.15). □

Theorem 3.3. *Let λ_0 be the largest real zero of the test function (3.14). If $\limsup_{n\to\infty}\sqrt[n]{\|\Delta_n\|} = h$ and $r \ge 2\ln\lambda_0$, then $\lim_{n\to\infty}\|\delta_n^Z f\| = 0$ for each $f \in AP(D_r)$.*

Proof. It is similar to the proof of Theorem 3.2. Let

$$H_n(z) = (-2ie^{\frac{1}{2}iz})^n\Delta_n(z), \qquad H_n^*(w) = H_n(-i\ln w). \tag{3.16}$$

By the lemma, take μ such that $h < \mu$ and observe that

$$0 < \frac{2he^r - 1}{e^r - 2h} < \frac{2\mu e^r - 1}{e^r - 2\mu} < e^{\frac{r}{2}}. \tag{3.17}$$

Let

$$\sigma_n(w) = \frac{\sqrt{H_n^*(w)}}{2\mu}. \tag{3.18}$$

Now we get in the same way as above

$$\sqrt[n]{\|\Delta_n^{-1}\|_r} \le \frac{2\mu e^r - 1}{\mu(e^r - 2\mu)}e^{-\frac{r}{2}}, \tag{3.19}$$

and finally,

$$\|\delta_n^Z f\| \le O(1)\left[\frac{2\mu e^r - 1}{e^{\frac{r}{2}}(e^r - 2\mu)}\right]^n \to 0 \quad \text{as } n \to \infty. \quad \square$$

Remark 4. We point out that Theorem 3.3 is more powerful than Theorem 3.2 although the latter is more convenient for application. In fact, we may prove $\lambda_0^2 \le 4h^2 + \sqrt{16h^4 - 1}$ where λ_0 is the largest zero of the test function (3.14). It is easy to see that

$$\omega(\lambda^2) = \Omega(\lambda)(\lambda + 2h) - 2h(2h-1)\left(\lambda - \frac{1}{2h}\right)(\lambda - 1),$$

so $\omega(\lambda_0^2) \le 0$ since $\lambda_0^2 \ge 2h \ge 1$; thus $\lambda_0^2 \le 4h^2 + \sqrt{16h^4 - 1}$ and the equality is taken if and only if $h = \frac{1}{2}$.

4. The estimate of $r(h)$

Definition 4.1. We define a function $r = r(h)$ such that if $R > r$ then

$$\lim_{n\to\infty} \|\delta_n^Z f\| = 0 \tag{4.1}$$

for $f \in AP(D_R)$ and each sequence of nodal sets Z_n satisfying $\limsup_{n\to\infty} \sqrt[n]{\|\Delta_n\|} = h$. Because of (2.9) we consider the function r only for $\frac{1}{2} \le h \le 1$.

By Lemmas 2.1, 3.1 and 3.2, we get the following result.

Theorem 4.1. *Let $r(h)$ be defined as above. Then*

$$\ln(4h - 1) \le r \le 2\ln(h + \sqrt{1 + h^2}), \quad \frac{1}{2} \le h \le h_0 \tag{4.2}$$

and

$$\ln(4h - 1) \le r \le \ln(4h^2 + \sqrt{16h^4 - 1}), \quad h_0 \le h \le 1, \tag{4.3}$$

where h_0 is the solution of equation $2\ln(h + \sqrt{1 + h^2}) = \ln(4h^2 + \sqrt{16h^4 - 1})$ in the interval $(\frac{1}{2}, 1]$.

Proof. The right inequalities in (4.2) and (4.3) are the results of Theorems 3.1 and 3.2 respectively.

It is enough to check the rest for a special sequence of nodal sets. But by the second part of Lemma 2.1, we have

$$h \le \frac{e^r + 1}{4},$$

and thus

$$r \ge \ln(4h - 1),$$

that is the left inequalities of both (4.2) and (4.3). □

For a given sequence of nodal sets $\Delta_n : t_{n,1} < t_{n,2} < \cdots < t_{n,n}$ $(0 \le t_{n,1}, t_{n,n} < 2\pi)$, define

$$h_n := \sqrt[n]{\sup_{t\in[0,2\pi]} \prod \left|\sin \frac{t - t_{n,j}}{2}\right|}, \tag{4.4}$$

$$\eta_{r,n} := \sqrt[n]{\min_{\operatorname{Im} z=r} \prod \left|\sin \frac{z - t_{n,j}}{2}\right|} \tag{4.5}$$

and let

$$h = \limsup_{n\to\infty} h_n, \qquad \eta_r = \liminf_{n\to\infty} \eta_{r,n}. \tag{4.6}$$

Let D_r° denote the inner of rectangle D_r.

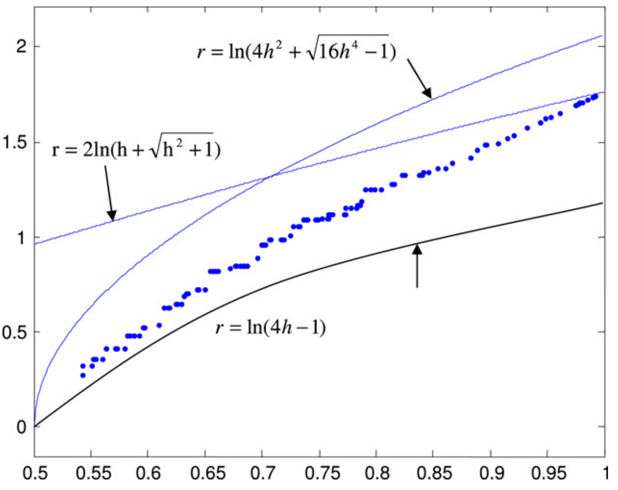

Fig. 1. Picture of $r(h)$.

Theorem 4.2. *Suppose that h, η_r and $t_{n,j}$ satisfy (4.4)–(4.6); then*

a. *If $h < \eta_r$, then $\lim_{n\to\infty} \|\delta_n^Z f\| \to 0$ for each $f(z) \in AP(D_r)$.*
b. *If $h > \eta_r$, then there exists $f(z) \in AP(D_r^\circ)$ such that $\lim_{n\to\infty} \|\delta_n^Z f\| \neq 0$.*

Proof. Without loss of the generality, let n be an even integer.

a. By (3.2), we have

$$\begin{aligned} |(\delta_n^Z f)(t)| &= \left| \frac{\Delta_n(t)}{4\pi \mathrm{i}} \int_{\partial D_r} \frac{f(z)}{\Delta_n(z)} \cot\frac{z-t}{2} \mathrm{d}z \right| \\ &= \left| \frac{\Delta_n(t)}{4\pi \mathrm{i}} \int_{\partial D_r^+} \frac{f(z)}{\Delta_n(z)} \cot\frac{z-t}{2} \mathrm{d}z + \frac{\Delta_n(t)}{4\pi \mathrm{i}} \int_{\partial D_r^-} \frac{f(z)}{\Delta_n(z)} \cot\frac{z-t}{2} \mathrm{d}z \right| \\ &\le \sup_{t\in[0,2\pi]} \left| \int_{\partial D_R^+ + \partial D_R^-} f(z) \cot\frac{z-t}{2} \mathrm{d}z \right| \frac{1}{4\pi} \left(\frac{h}{\eta_r}\right)^n \to 0. \end{aligned} \tag{4.7}$$

b. Let $\{n_k\}$ be a subsequence of $\{n\}$ such that $\eta_r = \lim_{n_k\to\infty} \eta_{r,n_k}$. Since $\{H_{n_k}\}$ is a normal family, there exists a converging subsequence of $\{H_{n_k}\}$ which is still denoted by H_n. So we can choose $z \in \mathbb{C}$ such that $\operatorname{Im} z_0 = r$ and $\lim_{n\to\infty} \sqrt[n]{|\Delta_n(z_0)|} = \eta_r$. Now we consider the interpolation of $f(z) = \cot\frac{z_0 - z}{2}$ by trigonometric polynomials. By [3]

$$\left[\frac{\Delta_n(t)}{\Delta_n(z_0)} - 1\right] \cot\frac{t - z_0}{2} = [\Delta_n(t) - \Delta_n(z_0)] \cot\frac{t - z_0}{2} \tag{4.8}$$

is a trigonometric polynomial. But

$$(T_n^Z f)(t_{n,j}) = f(t_{n,j}) = \left[\frac{\Delta_n(t_n,j)}{\Delta_n(z_0)} - 1\right] \cot\frac{t_{n,j} - z_0}{2},$$

and thus $[\frac{\Delta_n(t)}{\Delta_n(z_0)} - 1] \cot\frac{t - z_0}{2}$ is also a trigonometric interpolation polynomial of f. Since the trigonometric interpolation polynomial of f is unique, we have

$$(\delta_n^Z f)(t) = \frac{\Delta_n(t)}{\Delta_n(z_0)} \cot\frac{z_0 - t}{2}. \tag{4.9}$$

If $\lim_{n\to\infty} \|\delta_n^{\Delta} f\| = 0$, (4.9) gives

$$\lim_{n\to\infty} \frac{h_n}{|\Delta_n(z_0)|} = 0.$$

So there exists $N \in \mathbb{N}$ such that for $n > N$

$$h_n \le |\Delta_n(z_0)| \tag{4.10}$$

which is contracted to $h > \eta_r$. The proof is completed. □

There is very little information on the function $r(h)$ in the references currently, as far as we know. In the remainder of this article, we give pictures of this function by direct computation.

By Theorem 4.2, we can approximate $r(h)$ by using many sequences of nodal sets. For a given sequence of nodal sets $t_{n,j}$, we compute h and η_r. We choose about 100 sequences of nodal sets randomly in order to approximate $r_0 \in [\frac{1}{2}, 1]$ such that $\eta_{r_0} = h$. Finally, the points (h, r) are given in Fig. 1as $r(h)$, surrounded by $r = \ln(4h - 1)$, $r = 2\ln(h + \sqrt{h^2 + 1})$ and $r = \ln(4x^2 + \sqrt{16x^4 - 1})$ by Theorem 4.1.

Conjecture. *We conjecture that $r(h)$ is a continuous and monotonically increasing function.*

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